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Approximation of an Entire Function

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Let $f(x)$ be a real valued continuous function on $[-1, 1]$ and let

$$E_n(f) \equiv \inf_{p\in\pi_n} \|f - p\|, \qquad n = 0, 1, 2...,$$

where the norm is the maximum norm on $[-1, 1]$ and π_n denotes the set of all polynomials with real coefficients of degree at most n. Bernstein ([1], p. 118) proved that

$$\lim_{n\to\infty} E_n^{1/n}(f) = 0 \tag{1}$$

if, and only if, $f(x)$ is the restriction to $[-1, 1]$ of an entire function.

Let $f(z)$ be an entire function, and let

$$M(r) = M_f(r) = \max_{|z|=r} |f(z)|;$$

then the order ρ, lower order λ, type τ and lower type ω of $f(z)$ are defined by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log r} = \begin{matrix}\rho\\ \lambda\end{matrix} \qquad (0 \leqslant \lambda \leqslant \rho \leqslant \infty),$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\rho} = \begin{matrix}\tau\\ \omega\end{matrix} \qquad (0 \leqslant \omega \leqslant \tau \leqslant \infty) \tag{2}$$

(For the definitions of τ and ω, we require that $0 < \rho < \infty$).

Bernstein ([1], p. 114) has shown that there exists a constant $\rho > 0$ such that

$$\limsup_{n\to\infty} n^{1/\rho} E_n^{1/n}(f) \tag{3}$$

is finite if, and only if, $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ and some finite type τ.

Recently, Varga ([8], Theorem 1) has proved that

$$\lim_{n\to\infty} \sup \frac{n \log n}{\log[1/E_n(f)]} = \rho, \tag{4}$$

where ρ is a nonnegative real number if, and only if, $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ.

In Theorems 2 and 4, we extend the above results of Bernstein and Varga to the lower order and lower type of an entire function. We also give a short proof of Varga's theorem.

The above results suggest that the rate at which $E_n^{1/n}(f)$ tends to zero depends on the order and type of the entire function f. If an entire function is either of order $\rho = 0$ or of order $\rho = \infty$, then we cannot expect satisfactory results similar to (3) and (4). We shall deal, in Section I, with the case $\rho = \infty$ by assuming that there exists a positive integer $k \geqslant 2$, for which

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_{k+1}M(r)}{l_1 r} = \begin{matrix}\rho(k)\\ \lambda(k)\end{matrix} \tag{5}$$

are finite and positive. Here we have used the familiar notation

$$l_k x = \log\log \cdots (k \text{ times})\, x, \qquad (k = 1, 2, 3,...).$$

Note that $l_k x > 0$ for all sufficiently large positive x. An entire function $f(z)$ with $\rho(k-1) = \infty$ and $\rho(k) < \infty$ is called an entire function of index k. Thus, $\rho(k)$ and $\lambda(k)$ extend the definitions of ρ and λ in (2), which correspond to $k = 1$. If $\rho(k)$ is positive and finite, we can, as usual, associate with it functionals $\tau(k, f) = \tau(k)$ and $\omega(k, f) = \omega(k)$, defined by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_k M(r)}{r^{\rho(k)}} = \begin{matrix}\tau(k)\\ \omega(k)\end{matrix}. \tag{6}$$

The object of Section I is to study the relationship of $\rho(k)$ and $\tau(k)$ with the rate of growth of $E_n^{1/n}(f)$. Finally, we obtain the results of Bernstein and Varga as special cases of Theorems 1 and 3.

In Section II, a classification is introduced for the class of all entire functions of order $\rho = 0$, by means of the logarithmic order ρ_l and the corresponding logarithmic lower order λ_l. They are defined by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log\log r} = \begin{matrix}\rho_l\\ \lambda_l\end{matrix}. \tag{7}$$

This leads to theorems analogous to those valid for ρ. If ρ_l is larger than one and finite, we can define the logarithmic type of f, τ_l, and the corresponding lower type, ω_l by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{\rho_l}} = \begin{matrix}\tau_l\\ \omega_l\end{matrix}. \tag{8}$$

The main object of Section II is to investigate the relationship of the logarithmic order ρ_l and the corresponding logarithmic type τ_l with the asymptotic behavior of $E_n^{1/n}(f)$.

We need, for our purpose, the following lemma

LEMMA 1. ([4], Theorems 1 and 4(b)). *A necessary and sufficient condition that $f(z) = \sum_0^\infty a_n z^n$ be an entire function of index k, is that*

$$\lim_{n\to\infty} \sup \frac{n l_k n}{\log | 1/a_n |} = \rho(k).$$

LEMMA 2A. ([7], Theorem 1A) *Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of index k. Then*

$$\lim_{n\to\infty} \inf \frac{n l_k n}{\log | 1/a_n |} \leqslant \lambda(k).$$

LEMMA 2B. ([7], Theorem 1B) *Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of index k, such that $| a_{n-1}/a_n |$ is nondecreasing for $n > n_0$. Then*

$$\lim_{n\to\infty} \inf \frac{n l_k n}{\log | 1/a_n |} \geqslant \lambda(k).$$

LEMMA 3. ([6], Theorems 2 and 5) *A necessary and sufficient condition that $f(z) = \sum^\infty a_n z^n$ be an entire function of index k, with $\rho(k) > 0$, is that*

$$\lim_{n\to\infty} \sup \frac{n}{\rho e} | a_n |^{\rho/n} = \tau,$$

and

$$\lim_{n\to\infty} \sup (l_{k-1} n) | a_n |^{\rho(k)/n} = \tau(k), \qquad k = 2, 3, \ldots .$$

LEMMA 4A. ([7], Theorem) *Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of index k, with $\rho(k) > 0$. Then*

$$\lim_{n\to\infty} \inf \frac{n}{\rho e} | a_n |^{\rho/n} \leqslant \omega,$$

and

$$\lim_{n\to\infty} \inf (l_{k-1} n) | a_n |^{\rho(k)/n} \leqslant \omega(k), \qquad k = 2, 3, \ldots .$$

LEMMA 4B. ([7]) *Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of index k, with $\rho(k) > 0$, such that $|a_{n-1}/a_n|$ is nondecreasing for $n > n_0$. Then*

$$\liminf_{n\to\infty} \frac{n}{\rho e} |a_n|^{\rho/n} \geqslant \omega,$$

and

$$\liminf_{n\to\infty} (l_{k-1}n) |a_n|^{\rho(k)/n} \geqslant \omega(k), \qquad k = 2, 3, 4, \dots .$$

LEMMA 5. ([4], Theorems 1 and 3) *A necessary and sufficient condition that $f(z) = \sum_0^\infty a_n z^n$ be an entire function of finite logarithmic order ρ_l (which is necessarily $\geqslant 1$), in that*

$$\limsup_{n\to\infty} \frac{\log n}{\log\{1/n \log |1/a_n|\}} = \rho_l - 1.$$

LEMMA 6A. ([4], Theorem 3A) *Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of finite logarithmic lower order λ_l (necessarily $\geqslant 1$). Then*

$$\liminf_{n\to\infty} \frac{\log n}{\log\{1/n \log |1/a_n|\}} \leqslant \lambda_l - 1.$$

LEMMA 6B. ([4], Theorem 3B) *Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of finite logarithmic lower order λ_l, such that $|a_{n-1}/a_n|$ is nondecreasing for $n > n_0$. Then*

$$\liminf_{n\to\infty} \frac{\log n}{\log\{1/n \log |1/a_n|\}} \geqslant \lambda_l - 1.$$

LEMMA 7. ([5], Theorems 1, 2) *A necessary and sufficient condition that $f(z) = \sum_0^\infty a_n z^n$ be an entire function of logarithmic order ρ_l $(1 < \rho_l < \infty)$, is that*

$$\limsup_{n\to\infty} \frac{\{n/\rho_l\}^{\rho_l}}{\left\{\dfrac{\log |1/a_n|}{\rho_l - 1}\right\}^{\rho_l - 1}} = \tau_l .$$

LEMMA 8A. ([5], Theorems 1A and 2A) *Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of logarithmic order ρ_l $(1 < \rho_l < \infty)$. Then*

$$\liminf_{n\to\infty} \frac{\{n/\rho_l\}^{\rho_l}}{\left\{\dfrac{\log |1/a_n|}{\rho_l - 1}\right\}^{\rho_l - 1}} \leqslant \omega_l .$$

LEMMA 8B. ([5], Theorems 1A and 2B) *Let* $f(z) = \sum_0^\infty a_n z^n$ *be an entire function of logarithmic order* ρ_l $(1 < \rho_l < \infty)$, *such that* $|a_{n-1}/a_n|$ *is non-decreasing for* $n > n_0$. *Then*

$$\liminf_{n\to\infty} \frac{\{n/\rho_l\}^{\rho_l}}{\left\{\dfrac{\log|1/a_n|}{\rho_l - 1}\right\}^{\rho_l - 1}} \geqslant \omega_l .$$

SECTION I

THEOREM 1. *Let* $f(x)$ *be a real valued continuous function on* $[-1, 1]$ *and let* k *be a positive integer. Then*

$$\limsup_{n\to\infty} \frac{n l_k n}{l_1[1/E_n(f)]} = \sigma \tag{9}$$

satisfies $0 < \sigma < \infty$ *if, and only if,* $f(x)$ *is the restriction to* $[-1, 1]$ *of an entire function of index* k, *with* $\rho(k) = \sigma$.

Proof. First, assume that $f(x)$ has an entire extention $f(z)$ of index K with $0 < \sigma = \rho(k) < \infty$. Following Bernstein's original proof, we have ([2], p. 78), for each $n \geqslant 0$,

$$E_n(f) \leqslant \frac{2B(\sigma)}{\sigma^n(\sigma - 1)} \qquad \text{for every} \quad \sigma > 1, \tag{10}$$

where $B(\sigma)$ is the maximum of the absolute value of $f(z)$ on E_σ, and E_σ $(\sigma > 1)$ denotes the closed interior of the ellipse with foci at ± 1, major semi-axis $\sigma^2 + 1/2\sigma$ and minor semi-axis $\sigma^2 - 1/2\sigma$. Then,

$$D_1(\sigma) \equiv \left\{z \,\middle|\, |z| \leqslant \frac{\sigma^2 - 1}{2\sigma}\right\} \subset E_\sigma \subset D_2(\sigma) \equiv \left\{z \,\middle|\, |z| \leqslant \frac{\sigma^2 + 1}{2\sigma}\right\}.$$

From this inclusion, it follows by definition that

$$M_f\left(\frac{\sigma^2 - 1}{2\sigma}\right) \leqslant B(\sigma) \leqslant M_f\left(\frac{\sigma^2 + 1}{2\sigma}\right) \qquad \text{for all} \quad \sigma > 1. \tag{11}$$

From this, one can verify easily, for $k = 1, 2,..., j = 0, 1, 2,...,$ that

$$\begin{matrix}\rho(k,j) \\ \lambda(k,j)\end{matrix} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{l_{k+j} M(\sigma)}{l_{j+1}\sigma} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{l_{k+j} B(\sigma)}{l_{j+1}\sigma} . \tag{12}$$

The numbers $\rho(k,j)$, $\lambda(k,j)$ defined by (12), satisfy

$$\frac{\rho(1,1)}{\lambda(1,1)} = \frac{\rho_l}{\lambda_l}, \qquad \frac{\rho(k,0)}{\lambda(k,0)} = \frac{\rho(k)}{\lambda(k)}$$

$$\frac{\rho(2,0)}{\lambda(2,0)} = \frac{\rho}{\lambda}.$$

From (10), we have

$$E_n(f) \leqslant CB(\sigma)/\sigma^n, \tag{13}$$

where $C = 2/\sigma - 1$. From (13), we obtain for each $\eta > 0$,

$$\sum_{k=0}^{\infty} E_k(f)\,\sigma^k \leqslant \sum_{k=0}^{\infty} C\frac{B(\sigma+\eta)}{(\sigma+\eta)^k}\sigma^k = CB(\sigma+\eta)\sum_{k=0}^{\infty}\left(\frac{\sigma}{\sigma+\eta}\right)^k$$
$$\leqslant \frac{CB(\sigma+\eta)(\sigma+\eta)}{\eta}. \tag{14}$$

It is known ([8], (12) that

$$B(\sigma) = E_0 + 2\sigma\sum_{k=0}^{\infty} E_k\sigma^k, \tag{15}$$

where E_k is a nonincreasing sequence of real numbers. Consider the entire function

$$H(\sigma) = \sum_{k=0}^{\infty} E_k\sigma^k. \tag{16}$$

We have, from (14) and (15),

$$B(\sigma) \leqslant C'\sigma H(\sigma) \leqslant C''\sigma(\sigma+\eta)\,B(\sigma+\eta), \tag{17}$$

where C', C'' are some constants. From (12) and (17) we can verify that

$$\frac{\rho(k,j)}{\lambda(k,j)} = \lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{l_{k+j}B(\sigma)}{l_{j+1}\sigma} = \lim_{\sigma\to\infty}\frac{\sup}{\inf}\frac{l_{k+j}H(\sigma)}{l_{j+1}\sigma}. \tag{18}$$

Applying Lemma 1 to $H(\sigma)$, we obtain the required result (9).

Remark. For $k = 1$, Theorem 1 gives Varga's result.

THEOREM 2A. *Let $f(x)$ be a real valued continuous function on* $[-1, 1]$. *If $f(x)$ is the restriction to* $[-1, 1]$ *of an entire function of index k, then*

$$\liminf_{n\to\infty}\frac{nl_kn}{\log[1/E_n(f)]} \leqslant \lambda(k). \tag{19}$$

Proof. From (12) and (18), we have

$$\liminf_{\sigma\to\infty} l_{k+1}H(\sigma)/l_1\sigma = \lambda(k).$$

Now, applying Lemma 2A to $H(\sigma)$, we have the required result.

THEOREM 2B. *If $f(x)$ is the restriction to $[-1, 1]$ of an entire function of index k, and if $E_{n-1}(f)/E_n(f)$ is nondecreasing for $n > n_0$, then*

$$\liminf_{n\to\infty} \frac{nl_k n}{\log[1/E_n(f)]} \geqslant \lambda(k). \tag{20}$$

Proof. From (12) and (18), we have

$$\lambda(k) = \liminf_{\sigma\to\infty} l_{k+1}H(\sigma)/l_1\sigma.$$

Applying Lemma 2B to $H(\sigma)$, we have (20).

THEOREM 3. *Let $f(x)$ be a real valued continuous function on $[-1, 1]$, and let k be a positive integer. Then*

$$\begin{aligned} &\limsup_{n\to\infty} nE_n^{\rho/n}(f) = (\rho\, e\, \tau)\, 2^{-\rho} \text{ and, if } \quad k > 1, \\ &\limsup_{n\to\infty}(l_{k-1}n)\, E_n^{\rho(k)/n}(f) = \tau\,(k)\, 2^{-\rho(k)} \end{aligned} \tag{21}$$

are finite if, and only if, $f(x)$ is the restriction to $[-1, 1]$ of an entire function of index k, with $\rho(k) > 0$ and $\tau(k)$ finite.

Proof. We have, from (11) and (17),

$$2^{-\rho(k)} \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_k M(\sigma)}{\sigma^{\rho(k)}} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_k B(\sigma)}{\sigma^{\rho(k)}} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_k H(\sigma)}{\sigma^{\rho(k)}}. \tag{22}$$

Applying Lemma 3 to $H(\sigma)$, we obtain (21).

Remark. For $k = 1$, we obtain Bernstein's result on the finiteness of (3).

THEOREM 4A. *Let $f(x)$ be a real valued continuous function on $[-1, 1]$. If $f(x)$ is the restriction to $[-1, 1]$ of an entire function of index k, with $\rho(k) > 0$ and $\omega(k) > 0$, then*

$$\begin{aligned} &\liminf_{n\to\infty} nE_n^{\rho/n}(f) < \infty \qquad \text{and, if} \quad k > 1, \\ &\liminf_{n\to\infty}(l_{k-1}n)\{E_n(f\}^{\rho(k)/n} < \infty. \end{aligned} \tag{23}$$

Proof. We have from (22),

$$\liminf_{\sigma\to\infty} l_k H(\sigma)/\sigma^{\rho(k)} = \omega(k)\, 2^{-\rho(k)}.$$

Applying Lemma 4A to $H(\sigma)$, we have (23).

THEOREM 4B. *Let $f(x)$ be a real valued continuous function on* $[-1, 1]$. *If $f(x)$ is the restriction to* $[-1, 1]$ *of an entire function of index k, with $\rho(k) > 0$ and $\omega(k) > 0$, and if $E_{n-1}(f)/E_n(f)$ is nondecreasing for $n > n_0$, then*

$$\liminf_{n\to\infty} nE_n^{\rho/n}(f) > -\infty \qquad \text{and, if} \quad k > 1,$$
$$\liminf_{n\to\infty} (l_{k-1}n)\{E_n(f)\}^{\rho(k)/n} > -\infty. \tag{24}$$

Proof. This follows from (22), by applying Lemma 4B to $H(\sigma)$.

Second Proof of Varga's Theorem. A proof of this theorem can be carried out exactly like that of Theorem 1 of Okamura ([3], p. 133), but with one difference. In our proof, we use the inequality $E_n(f)\,\sigma^n \leqslant CB(\sigma)$ together with (12), while Okamura uses the inequality $|a_n|\, r^n \leqslant M(r)$ and the definition of order of an entire function.

SECTION II

THEOREM 5. *Let $f(x)$ be a real valued continuous function on* $[-1, 1]$. *Then*

$$\limsup_{n\to\infty} \frac{\log n}{\log\{(1/n)\log[1/E_n(f)]\}} = \alpha \tag{25}$$

satisfies $0 \leqslant \alpha < \infty$ if, and only if, $f(x)$ is the restriction to $[-1, 1]$ *of an entire function of logarithmic order $\rho_l = 1 + \alpha$.*

Proof. We have, from (18),

$$\rho_l = \rho(1, 1) = \limsup_{\sigma\to\infty} l_2 H(\sigma)/l_2\sigma.$$

Applying Lemma 5 to $H(\sigma)$, we obtain (25).

THEOREM 6A. *If $f(x)$ is the restriction to* $[-1, 1]$ *of an entire function of logarithmic lower order λ_l, where λ_l is a finite number (necessarily $\geqslant 1$), then*

$$\liminf_{n\to\infty} \frac{l_1 n}{l_1\{1/n\, l_1[1/E_n(f)]\}} \leqslant \lambda_l - 1. \tag{26}$$

Proof. We have, from (18),

$$\lambda(1, 1) = \lambda_l = \lim_{\sigma\to\infty} \inf l_2 H(\sigma)/l_2\sigma.$$

Applying Lemma 6A to $H(\sigma)$, we have the required result.

THEOREM 6B. *Let $f(x)$ be a real valued continuous function on $[-1, 1]$, which is the restriction to $[-1, 1]$ of an entire function of logarithmic lower order λ_l, $\lambda_l \geqslant 1$, and let $E_{n-1}(f)/E_n(f)$ be nondecreasing for $n > n_0$. Then*

$$\lim_{n\to\infty} \inf \frac{l_1 n}{l_1\{1/n\, l_1[1/E_n(f)]\}} \geqslant \lambda_l - 1. \tag{27}$$

Proof. Applying Lemma 6B to $H(\sigma)$, we have the required result.

THEOREM 7. *Let $f(x)$ be a real valued continuous function on $[-1, 1]$. If $f(x)$ is the restriction to $[-1, 1]$ of an entire function of logarithmic order $\rho_l > 1$, with $\tau_l \geqslant 0$, then*

$$\lim_{n\to\infty} \sup \frac{\{n/\rho_l\}^{\rho_l}}{\{-\log E_n(f)/(\rho_l - 1)\}^{\rho_l - 1}} \tag{28}$$

is finite.

Proof. From (11) and (17), observing that $1 < \rho_l < \infty$, we have

$$\begin{matrix}\tau_l \\ \omega_l\end{matrix} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(\sigma)}{(\log \sigma)^{\rho_l}} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log B(\sigma)}{(\log \sigma)^{\rho_l}} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log H(\sigma)}{(\log \sigma)^{\rho_l}}. \tag{29}$$

Applying Lemma 7 to $H(\sigma)$, we have (28).

THEOREM 8. *Let $f(x)$ be a real valued continuous function on $[-1, 1]$. If $f(x)$ is the restriction to $[-1, 1]$ of an entire function of logarithmic order $\rho_l > 1$.* With $\omega_l > 0$, then

$$\lim_{n\to\infty} \inf \frac{\{n/\rho_l\}^{\rho_l}}{\{-l_1 E_n(f)/\rho_l - 1\}^{\rho_l - 1}} < \infty.$$

Proof. We have from (29).

$$\begin{matrix}\tau_l \\ \omega_l\end{matrix} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix}\, l_1 H(\sigma)/(l_1\sigma)^{\rho_l}.$$

Applying Lemma 8A to $H(\sigma)$, we have the result.

THEOREM 8B. *If $f(x)$ is the restriction to $[-1, 1]$ of an entire function of logarithmic order $\rho_l > 1$ and of finite logarithmic lower type ω_l, such that $E_{n-1}(f)/E_n(f)$ is nondecreasing for $n > n_0$, then*

$$\liminf_{n\to\infty} \frac{\{n/\rho_l\}^{\rho_l}}{\{-l_1 E_n(f)/(\rho_l - 1)\}^{\rho_l - 1}} \geqslant 0. \tag{30}$$

Proof. We have, from (29),

$$\omega_l = \liminf_{\sigma\to\infty} \log H(\sigma)/(\log \sigma)^{\rho_l}.$$

Applying Lemma 8B to $H(\sigma)$, we obtain (30).

Added in proof: Lemmas stated here are slightly different from the original sources.

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REFERENCES

1. S. N. BERNSTEIN, "Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle," Gauthier-Villars, Paris, 1926.
2. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart and Winston, New York, 1966.
3. H. OKAMURA, Sur les coefficients de Taylor des fonctions entieres d'ordre fini, *Tohoku Math. J.* **38** (1933), 129–144.
4. A. R. REDDY, On entire Dirichlet series of zero order, *Tohoku Math. J.* **16** (1966), 144–155.
5. A. R. REDDY, On entire Dirichlet series of zero order (II) (to appear).
6. A. R. REDDY, On entire Dirichlet series of infinite order (I), *Rev. Math. His. Amer.* **27** (1967), 120–131.
7. A. R. REDDY, On entire Dirichlet series of infinite order (II), *Rev. Math. His. Amer.* **29** (1969), 215–231.
8. R. S. VARGA, On an extention of a result of S. N. Bernstein, *J. Approx. Theory* **1** (1968), 176–179.